The Existence of Nonconstant Periodic Solutions to the Second-Order Discrete Equation

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Abstract: This article is concerned with the existence of nonconstant periodic solutions for nonlinear second-order difference equations. In this paper, the existence of nonconstant periodic solution is obtained by using the critical point theorem and variational frameworks. First, we introduce some appropriate variational frameworks. The existence of nonconstant periodic solutions is equivalent to the existence of critical points of the functional. Second, using critical point theorem, we obtain some critical points, the nonconstant periodic solutions are obtained. The work replenishes a blank of this part.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the set of all natural numbers, integers and real numbers respectively. For a, b $\in\mathbb{Z}$, define $\mathbb{Z}(a)=\{a,a+1,a+2,\cdots\}$, when $a\le b$, define $\mathbb{Z}(a,b)=\{a,a+1,a+2,\cdots,b\}$.

Consider the discrete system,

$$\Delta\left(p_n\left(\Delta x_{n-1}\right)^{\delta}\right)+f\left(n,x_n\right)=0, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$, $\Delta^2 x_n=\Delta\left(\Delta x_n\right)$, and $(-1)^{\delta}=-1$, $\delta>0$, $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is continuous in the second variable , and $F(n,z)$ is defined as $F(n,z)=\int_0^z f(n,s)\,ds$, $p_{T+n}=p_n>0$, for some integer T . In this paper, we let $v_1=\min_{n\in\mathbb{Z}(1,T)}\{p_n\}>0$, $v_2=\max_{n\in\mathbb{Z}(1,T)}\{p_n\}>0$.

We may think of equation (1) as being a discrete analogue of a special case of the second order differential equation

$$\left(p(t)\varphi(u')\right)'=f(t,u), \tag{2}$$

which has been studied by many authors [5, 8, 9]. In the case of $\varphi(u)=|u|^{\delta-2}u$, equation (2) has been discussed extensively in the literature, we may refer to [9, 10, 12, 13]. When $\delta=1$ and $f(n,u)=q_n u$, equation (1) has been investigated by many authors for results on oscillation, asymptotic behavior and boundary value problems [2, 3, 6, 7]. But the results on existence of periodic solutions of nonlinear difference equations are very scare in the literature [4, 15].

Cai Yu and Guo in [4] obtained some new sufficient conditions for the existence of nontrivial solution of equation (1) via critical point method. In that paper, they need such a condition $\lim_{|z|\to\infty}\frac{f(n,z)}{z^{\delta}}=+\infty$ and $\lim_{|z|\to 0}\frac{f(n,z)}{z^{\delta}}=0$. Recently,Hongxia Tang and Limei Wei in [1] get some solution when $\lim_{|z|\to\infty}\frac{f(n,z)}{z^{\delta}}=0$ is satisfied.

The main purpose in our paper is to establish the variational framework of equation (1) in order to get some nonconstant T-periodic solution when $\lim_{|z|\to\infty}\frac{f(n,z)}{z^{\delta}}=0$ is satisfied.

The main results are as follows:

Theorem 1.1 Suppose $F(n,z)$ satisfies

(C1) $F(n,z)\in C(\mathbb{R},\mathbb{R})$ for each $n\in\mathbb{Z}$ and there exists a positive integer T such that for all $(n,z)\in\mathbb{Z}\times\mathbb{R}$, $F(n+T,z)=F(n,z)$.

(C2) there exists a constant $\alpha\in(1,\delta+1)$ such that

$$0<z\cdot f(n,z)\le\alpha F(n,z),\forall(n,z)\in(\mathbb{Z}\times\mathbb{R}) \text{ and } |z|\neq 0, \tag{3}$$

(C3) there exist constants $a_5>0$ and $\gamma\in(1,\alpha]$ such that

$$F(n,z)\ge a_5|z|^{\gamma},\forall(n,z)\in(\mathbb{Z}\times\mathbb{R}). \tag{4}$$

Then equation (1) possesses at least one nonconstant T-periodic solution.

2. Some basic lemmas

In order to apply the critical point theory, we introduce some appropriate variational frameworks in this section.

Let S be the set of sequences $x=(\cdots,x_{-n},\cdots,x_{-1},x_0,x_1,\cdots,x_n,\cdots)=\{x_n\}_{n=-\infty}^{+\infty}$, i.e., $S=\{x=\{x_n\}\mid x_n\in\mathbb{R},n\in\mathbb{Z}\}$. For any $x,y\in S$, $a,b\in\mathbb{R}$, $ax+by$ is defined by $ax+by:=\{ax_n+by_n\}$, then S is a vector space.

For any given positive integer T, E_T is defined as a subspace of S by

$$E_T=\{x=\{x_n\}\in S:x_{n+T}=x_{n,}n\in\mathbb{Z}\}.$$

We note that E_T can be equipped with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ as follows:

$$(x,y)=\sum_{j=1}^{T}x_j\cdot y_j,\forall x,y\in E_T, \tag{5}$$

$$\|x\|=\left(\sum_{j=1}^{T}x_j^2\right)^{\frac{1}{2}},\forall x\in E_T, \tag{6}$$

It is obvious that E_T with the inner product in the (5) is a finite-dimensional Hilbert space and linearly homeomorphic to R^T.

We define the functional J on E_T as follows:

$$J(x)=\frac{1}{\delta+1}\sum_{n=1}^{T}p_{n+1}(\Delta x_n)^{\delta+1}-\sum_{n=1}^{T}F(n,x_n). \tag{7}$$

It is easy to see that $J\in C^1(E_T,\mathbb{R})$ and for any $x\in E_T$, by using $x_T=x_0$, $x_{T+1}=x_1$ we can compute the partial derivative as

$$\frac{\partial J}{\partial x_n} = -\Delta\left[p_n\left(\Delta x_{n-1}\right)^{\delta}\right] - f\left(n,x_n\right), n \in \mathbb{Z}\left(1,T\right)$$

then x is a critical point of J on E_T if and only if

$$\Delta\left(p_n\left(\Delta x_{n-1}\right)^{\delta}\right) + f\left(n,x_n\right) = 0$$

By the periodicity of x_n and $f\left(n,z\right)$ in the first variable n, we have reduced the existence of periodic solutions of equation (1) to the existence of critical points of J on E_T . For convenience, we identify $x \in E_T$ with $x = \left(x_1, x_2, \cdots, x_T\right)^T$.

Denote $W = \left\{x \in E_T : x_i = v, v \in \mathbb{R}, i \in \mathbb{Z}\left(1,T\right)\right\}$ and $W^{\perp} = Y$, such that $E_T = W \oplus Y$. Denote the norm $\|\cdot\|_r$ on E_T as follows: $\|x\|_r = \left(\sum_{i=1}^{T}\left|x_i\right|^r\right)^{\frac{1}{r}}$, for all $x \in E_T$ and $r > 1$. Clearly, $\|x\| = \|x\|_2$. Since || · ||r and || · || are equivalent, so it's easy to get

$$T^{-1}\|x\|_r \le \|x\| \le T\|x\|_r , \forall x \in E_T , \tag{8}$$

Let X be a real Banach space, $I \in C^1\left(X,\mathbb{R}\right)$, that is, I is a continuously Frechet differentiable functional defined on X . The functional I is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u_n\right\} \subset X$, for which $\left\{I\left(u_n\right)\right\}$ is bounded and when $n \to \infty$, $I'\left(u_n\right) \to 0$, then $\left\{u_n\right\}$ possesses a convergent subsequence in X .

Let B_r denote the open ball in X about 0 of radius r and let ∂B_r denote its boundary.

Lemma 2.1 (Saddle point theorem [11]) Let X be a real Banach space, $X = X_1 \oplus X_2$ where $X_1 \ne \left\{0\right\}$ and is finite dimensional. Suppose $I \in C^1\left(X,\mathbb{R}\right)$ satisfies the P-S condition and

(I1) there exist constants $\sigma > 0$ and $\rho > 0$ such that $I\mid_{\partial B_\rho \cap X_1} \le \sigma$;

(I2) there exist $e \in B_\rho \cap X_1$ and a constant $\omega > \sigma$ such that $I\mid_{e+X_1} \ge \omega$.

Then I possesses a critical value $c \ge \omega$ and $c = \inf_{h\in\Gamma}\max_{u\in B_\rho \cap X_1} I\left(h\left(u\right)\right)$, where

$$\Gamma = \left\{h \in C\left(\bar{B}_\rho \cap X_1, X\right) : h\mid_{\partial \bar{B}_\rho \cap X_1} = id\right\}.$$

3. The proofs of main results

PROOF OF THEOREM 1.1

According to the proof of Theorem 1.1, we know that the functional $J \in C^1\left(E_T,\mathbb{R}\right)$ has a critical value c and equation (1) has a corresponding solution. By (C1), we see that the zero solution is a constant solution of equation (1). If $x_n \equiv z \in \mathbb{R}$ is another constant solution of equation (1), then $f\left(n,z\right) = 0$. By (3), we have $z = 0$. Thus the zero solution is the only constant solution of system (1). Since $J\left(0\right) = 0$, in order to complete the proof it suffices to show that J has a critical value $c < 0$.

Assume that $y^0 \in Y$ and $\left\|y^0\right\| = \eta$ small enough, it holds that

$$\omega = \eta^{\gamma}\left(a_5\left(\frac{1}{T}\right)^{\gamma} - \frac{C}{\delta+1}\eta^{\delta+1-\gamma}\right) > 0 .$$

For any $x = y^0 + w, w \in W$, we have

$$
\begin{aligned}
-J(x) &= -\frac{1}{\delta+1}\sum_{n=1}^{T} p_{n+1}\left(\Delta x_n\right)^{\delta+1} + \sum_{n=1}^{T} F\left(n, x_n\right) \\
&= -\frac{1}{\delta+1}\sum_{n=1}^{T} p_{n+1}\left(\Delta y_n^0\right)^{\delta+1} + \sum_{n=1}^{T} F\left(n, x_n\right) \\
&\geq -\frac{v_2}{\delta+1}\sum_{n=1}^{T}\left(\Delta y_n^0\right)^{\delta+1} + \sum_{n=1}^{T} F\left(n, x_n\right) \\
&= -\frac{v_2}{\delta+1}\left\|\Delta y^0\right\|_{\delta+1}^{\delta+1} + \sum_{n=1}^{T} F\left(n, x_n\right) \\
&\geq -\frac{v_2}{\delta+1}\left\|T\Delta y^0\right\|^{\delta+1} + \sum_{n=1}^{T} F\left(n, x_n\right) \\
&\geq -\frac{C}{\delta+1}\left\|y^0\right\|^{\delta+1} + a_5\left(\frac{1}{T}\right)^{\gamma}\left(\left\|y^0\right\|^2 + \left\|w\right\|^2\right)^{\frac{\gamma}{2}} \\
&\geq -\frac{C}{\delta+1}\left\|y^0\right\|^{\delta+1} + a_5\left(\frac{1}{T}\right)^{\gamma}\left\|y^0\right\|^{\gamma} \\
&= \eta^{\gamma}\left(a_5\left(\frac{1}{T}\right)^{\gamma} - \frac{C}{\delta+1}\eta^{\delta+1-\gamma}\right) = \omega
\end{aligned}
$$

This show that $-J$ satisfies the condition (I2), where $C = v_2 T^{\delta+1}\lambda_T^{\frac{\delta+1}{2}}$. For any $y \in Y$

$$
\begin{aligned}
-J(y) &= -h(y) + \sum_{n=1}^{T} F\left(n, y_n\right) \\
&\leq -L\|y\|^{\delta+1} + \sum_{n=1}^{T}\left(a_3\left|y_n\right|^{\alpha} + a_4\right) \\
&\leq -L\|y\|^{\delta+1} + a_3\|y\|_{\alpha}^{\alpha} + a_4 T \\
&\leq -L\|y\|^{\delta+1} + a_3 T^{\alpha}\|y\|_2^{\alpha} + a_4 T \\
&\to -\infty .
\end{aligned}
$$

So there exists a sufficiently large constant $\rho > 0$ such that $-J(y) \leq \omega - 1$ for $y \in Y$, $\|y\| = \rho$. Which implies that J satisfies the condition (I1). Therefore $-J$ possesses a critical value $c \geq \omega > 0$. The proof of Theorem 1.1 is complete.

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References

[1] Hongxia Tang,Limei Wei. The Existence of Periodic Solutions to the Second-Order Discrete Equation[P]. 2021 3rd International Conference on Information Engineering for Mechanics and Materials,2021.

[2] Agarwal R.P., Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York, 2000.

[3] Ahlbrandt C.D. and Peterson A.C., Discrete Hamitonian Systems: Difference Equation, Continued Fractions, and Riccati Equation, Kluwer Academic Publishers, (1996).

[4] Cai X.C., Yu J.S. and Guo Z.M., Periodic Solutions of a Class of Nonliear Difference Equations via Critical Point Method, Comput. Math. Appl. 52 (2006) 1639-1647.

[5] Castro A. and Shivaji R., Nonnegetive Solutions to a Semilinear Dirivhlet Problem in a Ball are Positive ans Radially Symmetric, Comm. PDE. 14(8-9) (1989) (1091-1100).

[6] Chen S. and Erbe L.H., Riccati Techniques and Discrete Oscillations, J. Math. Anal. Appl. 142 (1989) 468-487.

[7] Chen S. and Erbe L.H., Oscillation and Nonoscillation for Systems of Self-adjoint Second-order Difference Equations, SIAM J. Math. Anal. 20 (1989) 939-949.

[8] Esteban J.R. and Vazguez J.L., On the Equation of Turbulent Filtration in One-dimensional Porous Media, Nonlinear Analysis 10(12)(1986)1303-1325.

[9] Herrero M.A. and Vazguez J.L., On the propagation properties of a nonliear degenerate parobolic equation, Comm Partial Differential Equations 7(12)(1982),1381-1402.

[10] Li W. T., Oscillation of certain Second-order Nonliear Differential Equations, J. Math. Anal. Appl. 217 (1998) 1-14.

[11] Marini M. and Zezza P., On the Asympototic Behavior of the Solutions of a Class of Second Order Linear Differential Equations, J. Diff. Equat. 28 (1978) 1-17.

[12] Rabinowitz P.H., Minimax Methods in Critical Point Theory with Application to Differential Equation, CBMS Reg. Conf. Ser. Math., vol.65, Amer. Math. Soc., Providence, RI, 1986.

[13] Wen L. Z., Asypototic and Oscillation of Second-order Functional Differential Equation, Science in China 2, (1986) 147-161, (in chinese).

[14] Wong P.J.Y. and Agarwal R.P., Oscillatory Behavior of Solutions certain Second Order Nonlinear Differencial Equations, J. Math. Anal. Appl. 198 (1996) 113-354.

[15] Zhou Z., Yu J.S. and Guo Z.M., The Existence of Periodic and Subharmonic Solutions to Subquadratic Discrete Hamiltonian Systems, ANZIAM J. 47 (2005) 89-102.